INVESTIGATION OF THE EQUATIONS FOR REGGE POLE PARAMETERS

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An existence and uniqueness theorem is proved for the solution of the system of equations for the Regge pole parameters proposed by Cheng, Sharp, and Mandelstam. The proof presupposes the fulfillment of a certain condition that relates all the parameters. The Kantorovich–Akilov theorem is used, fulfillment of the conditions of this theorem ensuring the contraction property of the mappings.

In 1963, Cheng and Sharp [1, 2] deduced an approximate system of equations for the Regge pole parameters $\alpha(t)$ and $\beta(t)$. Their derivation was based on the analyticity properties and unitarity of the amplitude of the two-particle reaction in the t channel.

It was assumed that the amplitude satisfies the Mandelstam representation and has a Regge behavior asymptotically. Under these conditions, $\alpha(t)$ and $\beta(t)/(q_1 q_1)^{\alpha(t)}$ were shown to be real analytic functions in the t plane with the cut $[t_0, \infty)$. An approximate unitarity condition for the partial waves was used and only the two-particle intermediate state and a single Regge pole were taken into account. The assumption of narrow resonances, valid for small t, was also made. These assumptions were used to derive a system of equations for the parameters $\alpha(t)$ and $\beta(t)$.

Similar equations for the case of scattering by a Yukawa potential were solved numerically and compared with the results obtained by the direct methods [3, 4]. The comparison showed that the results were, on the whole, similar. This prompted the idea that in the case of relativistic scattering, for which there are no direct methods, the system of equations of Cheng and Sharp point a way to a dynamical determination of the Regge parameters.

In 1968, Mandelstam [5] modified the equations of Cheng and Sharp in formulating a dynamical scheme based on linearly increasing Regge pole trajectories. However, Mandelstam [5] used only the trivial solution of these equations for $\mathrm{Im}\,\alpha(t) = 0$. An analogous system of equations was proposed at the same time by Epstein and Kaus [6]; they solved their system numerically for the case of the ρ-trajectory in the reaction $\pi^0 + \pi^0 \to \pi^0 + \pi^0$. Recently [7], corrections have been introduced into the system of equations of Epstein and Kaus [6]; these take into account the contribution of intermediate channels with more than two particles. However, a numerical calculation has shown that the results are only slightly affected [7].

This approach has also been used for bootstrap calculations of the Regge pole parameters [5, 6, 8].

The question of the existence and uniqueness of a solution of the system of equations for the Regge pole parameters was already proposed by Cheng and Sharp [1, 2]. However, to the best of our knowledge, this question remains open, both for the equations of [1, 2] as well as their subsequent modifications [5, 6, 7]. In the present paper, we consider the question of the existence and uniqueness of a solution of the system of equations of Cheng, Sharp, and Mandelstam with one subtraction. Methods of functional analysis are used to show that this system of equations has a unique solution if a certain condition which relates all the constants is satisfied.

We write the system of equations for $\alpha(t)$ and $\beta(t)$ in the form [5]

$$\alpha(t) = at + b + \frac{t}{\pi}\int_{t_0}^{\infty} dt' \frac{\mathrm{Im}\,\alpha(t')}{t'(t'-t)}, \tag{1}$$

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$$\beta(t)=\frac{E(t)(4q^2)^{\alpha(t)}}{(e/a)^{at+b}\Gamma(at+b+3/2)}\prod_{n=1}^{\infty}\frac{a(t-t_n)}{at+b+n+1/2}\exp\left\{-\frac{t}{\pi}\int_{t_0}^{\infty}dt'\frac{\operatorname{Im}\alpha(t')}{t'(t'-t)}\ln(4q'^2)\right\},\qquad(2)$$

$$\operatorname{Im}\alpha(t)=k(t)\beta(t),\quad t>t_0.\qquad(3)$$

Here, E(t) is an entire function of t; t_n are the values of t for which $\alpha(t)=-n-1/2$, $k(t)=\sqrt{(t-t_0)/t}$; and q^2 is the square of the momentum transfer.

The system (1)-(3) differs from the system of equations in Mandelstam's paper in that one subtraction has been introduced and that the exponent of (e/a) in Eq. (2) is $at+b$ and not $\alpha(t)$.

We reduce Eqs. (1)-(3) to a single equation; for values of t above the cut, we then have

$$\operatorname{Im}\alpha(t)=\frac{k(t)E(t)(4aq^2/e)^{at+b}}{\Gamma(at+b+3/2)}\prod_{n=1}^{\infty}\frac{a(t-t_n)}{at+b+n+1/2}\exp\left\{-\frac{t}{\pi}\int_{t_0}^{\infty}dt'\frac{\operatorname{Im}\alpha(t')}{t'(t'-t)}\ln\frac{q'^2}{q^2}\right\},\quad t>t_0.\qquad(4)$$

We note that $\operatorname{Im}\alpha(t)=0$ for $t<t_0$.

In contrast to (1) and (2), Eq. (4) is not singular since, as $t'\to t$,

$$\ln\frac{q'^2}{q^2}\Big/(t'-t)\to\frac{1}{t-t_0}.$$

For E(t) = const we have $\operatorname{Im}\alpha(t)\sim 1/l$ (if $t\to\infty$), which corresponds to the assumption of narrow resonances.

In Eq. (4), we make the change of variable

$$v=\frac{t-t_0}{t}.$$

Equation (4) then takes the form

$$\operatorname{Im}\alpha(v)=\Phi(v)\exp\left\{-\frac{1}{\pi}\int_0^1dv'\frac{\operatorname{Im}\alpha(v')}{v'-v}\ln\frac{q'^2}{q^2}\right\},\qquad(5)$$
$$0\leqslant v\leqslant 1.$$

Here,

$$\Phi(v)=\Phi(t)=\frac{Ek(t)(4aq^2/e)^{at+b}}{\Gamma(at+b+3/2)}\prod_{n=1}^{\infty}\frac{a(t-t_n)}{at+b+n+1/2}.\qquad(6)$$

Equation (5) has a number of interesting properties.

1. A solution of the equation is, if it exists, a positive function and

$$\operatorname{Im}\alpha(v)<\Phi(v).$$

2. The solution must vanish as $v^{\varkappa}$ in the limits $v\to 0$ and as $1-v$ in the limit $v\to 1$, where $\varkappa=at_0+b+1/2$. We introduce the new function

$$x(v)=\frac{\operatorname{Im}\alpha(v)}{v^{\varkappa/2}(1-v)^{1/2}}=\frac{\operatorname{Im}\alpha(v)}{\rho(v)}.\qquad(7)$$

Equation (5) takes the form

$$x(v)=\frac{\Phi(v)}{\rho(v)}\exp\left\{-\frac{1}{\pi}\int_0^1dv'\frac{\rho(v')x(v')}{v'-v}\ln\frac{q'^2}{q^2}\right\}\qquad(8)$$

or, in operator form,

$$x(v)=Px(v).\qquad(9)$$

Equations (5) and (8) are equivalent in the sense that each solution of one of them corresponds to a definite solution of the other.

In order to investigate the existence and uniqueness of a solution of Eq. (9), we shall use the methods of functional analysis. To be precise, we shall use a theorem proved in the book of Kantorovich and Akilov [9] (see also [10]).

THEOREM. Suppose an operation P maps a set $\Omega \subset X$ into X and at every point of a convex closed set $\Omega_0 \subset \Omega$ has a derivative. Then, if

$$P(\Omega_0) \subset \Omega_0, \tag{10}$$

$$\sup_{x \in \Omega_0} \|P'(x)\| < 1, \tag{11}$$

then there exists a unique fixed point of the operation P in Ω_0. If $P'(x)$ is continuous, the condition (11) is also necessary.

As the space X, we shall take the space $L_2(0, 1)$ of square-summable functions on [0, 1]. This imposes no additional restrictions on the desired function. As the sets Ω and Ω_0, we shall take the cone of negative functions in this space; this cone is a closed convex set.

As we have seen above, the condition (10) of the theorem is satisfied. In order to verify the condition (11), we must estimate the norm of the derivative P' of P.

The Fréchet differential of P has the form

$$P'(x)\,\delta x(v) = -\frac{\Phi(v)}{\rho(v)} \exp\left\{-\frac{1}{\pi}\int_0^1 dv' \frac{\rho(v')\,x(v')}{v'-v} \ln\frac{q'^2}{q^2}\right\} \frac{1}{\pi}\int_0^1 dv' \frac{\rho(v')\,\delta x(v')}{v'-v} \ln\frac{q'^2}{q^2}. \tag{12}$$

Let us make some estimates:

$$|P'(x)\,\delta x| \leqslant \frac{\Phi(v)}{\rho(v)} \frac{1}{\pi}\int_0^1 dv' \frac{\rho(v')\,\delta x(v')}{v'-v} \ln\frac{q'^2}{q^2} = \int_0^1 K(v, v')\,dv', \quad x \in \Omega_0, \tag{13}$$

$$\|P'(x)\,\delta x\| \leqslant \left(\int_0^1 dv \left[\int_0^1 K(v, v')\,dv'\right]^2\right)^{1/2}, \quad x \in \Omega_0,$$

$$\int_0^1 K(v, v')\,dv' = \frac{1}{\pi}\int_0^1 dv' \frac{\Phi(v)\rho(v)\rho(v')\,\delta x(v')}{\rho^2(v)(v'-v)} \ln\left(\frac{q'^2}{q^2}\right) = \frac{1}{\pi}\int_0^1 \frac{\Phi(v)}{\rho^2(v)} \frac{v^{1/2}(1-v)^{1/2} v'^{1/2}(1-v')^{1/2}}{(v'-v)} \ln\left(\frac{q'^2}{q^2}\right)$$

$$\times\, v^{(\varkappa-1)/2} v'^{(\varkappa-1)/2}\,\delta x(v')\,dv' = \frac{1}{\pi}\Psi(v)\int_0^1 \varphi(v, v')\, v^{(\varkappa-1)/2} v'^{(\varkappa-1)/2}\,\delta x(v')\,dv', \tag{14}$$

where $\Psi(v) = \Phi(v)/\rho^2(v)$ is a bounded function and

$$\varphi(v, v') = \frac{v^{1/2}(1-v)^{1/2} v'^{1/2}(1-v')^{1/2}}{v'-v} \ln\left(\frac{q'^2}{q^2}\right)$$

is also bounded and, as follows from a numerical calculation, its maximum does not exceed 1.2.

Then,

$$\|P'(x)\,\delta x\| \leqslant \frac{1}{\pi} \max\Psi \max\varphi \left(\int_0^1 dv \left[\int_0^1 dv' \times v^{(\varkappa-1)/2} v'^{(\varkappa-1)/2}\,\delta x'(v')\right]\right)^{1/2}.$$

Applying the Cauchy–Schwarz–Buniakowski inequality to the integral over v' and calculating the integrals, we obtain

$$\|P'(x)\,\delta x\| \leqslant \frac{\max\varphi \max\Psi}{\pi\varkappa} \|\delta x\|,$$

and hence

$$\|P'(x)\| \leqslant \frac{\max\varphi \max\Psi}{\pi\varkappa} \quad \text{for all} \quad x \in \Omega_0.$$

Let us estimate max $\Psi(v)$:

$$\Psi(t) = \frac{Ek(t)(4aq^2/e)^{at+b}}{\rho^2(t)\Gamma(at+b+3/2)}\prod_{n=1}^{\infty}\frac{a(t-t_n)}{at+b+n+1/2} \leqslant \frac{Ek(t)(4aq^2/e)^{at+b}}{\rho^2(t)\Gamma(at+b+3/2)},$$

since the infinite product does not exceed unity. Applying Stirling's formula for a positive argument to the Γ function, we find

$$\Psi(t) \leqslant \frac{E\sqrt{e}}{t_0 a\sqrt{2\pi}}.$$

Finally,

$$|P'(x)| \leqslant \frac{1.2E\sqrt{e}}{\pi^{1/4}t_0 a x} \simeq \frac{0.354E}{t_0 a x}. \tag{15}$$

A sufficient condition for (11) to hold is

$$0.354\frac{E}{t_0 a x} < 1, \quad x = at_0 + b + 1/2. \tag{16}$$

For the case of the ρ-trajectory, $a = 0.84$ (BeV/c)$^{-2}$, b = 0.57, t_0 = 0.072 (BeV/c)^2, E = 0.2 [6].

Substituting these values, we see that the condition (16) is satisfied for the ρ-trajectory.

Of course, the condition (16) is more stringent than is necessary for the existence of a unique solution. However, it is quite clear that a restriction of the type (16) on the trajectory parameters must hold for a solution to exist and to be unique; this circumstance must be taken into account in the construction of Regge bootstrap schemes and also in the determination of the actual solution of a system by the method of successive approximations.

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